Beam matching in the Sabina Undulator

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The SABINA undulator is composed by three UM modules as shown in Fig. 1. Each undulator is composed by 24 periods of 5.5 cm for a total magnetic length of 132 cm. The effective undulator length is $L_u = 135$ cm: at each side of the undulator "magic fingers" are installed, in order to minimize the first and the second field integrals. The magic fingers have a length of 2.1cm (at each side).

We analyze here the matching condition ensuring the minimum transverse beam size; the undulator is assumed in circular polarization for a beam energy comprised between 30 MeV and 100 MeV. The undulator has a maximum field strength K=3.4. The separation between two undulators L_d is assumed to be $L_d = 35$ cm, excluding the magic fingers. The matching is therefore calculated for an undulator of 132 cm with separations of $L_d = 39.2$ cm, (including the magic fingers in the separation length).

Figure 1: Schematic layout of the SABINA undulator line. The three modules have a length L_u and a separation (gap) L_d.

In circular polarization the undulator Apple-X has a symmetric focusing strength, the beam line doesn't include matching quadrupoles between the undulators, therefore matching in x and y directions is identical. For this reason we restrict the calculation to only one transverse plane. The Twiss parameters required are the same on both planes. The beam propagates with constant Twiss parameters (invariant envelope) along a single module when matched to the following Twiss parameters (addreference) :

$$T_M = \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}_M = \begin{pmatrix} \frac{\pi K}{\lambda_u \gamma} \\ 0 \\ \left(\frac{\pi K}{\lambda_u \gamma}\right)^{-1} \end{pmatrix} \tag{1}$$

This condition cannot be imposed at the entrance of the three undulators because of the lack of sufficient degrees of freedom along the beam line. If we impose matching at the entrance of the first module, the behavior of the $\beta_T(z)$is the one shown in Fig. 2. We have considered here a case at low energy $E = 30\, MeV$, targeting the longest wavelength in the range of SABINA, i.e. $\lambda_0 = 100\mu$m.

The beam traverses the first module with constant size, but in the gap between the first and the second module increases in size and enters mismatched in the second module. In the second module executes therefore a series of betatron oscillations up to the second gap, the one between the second and the

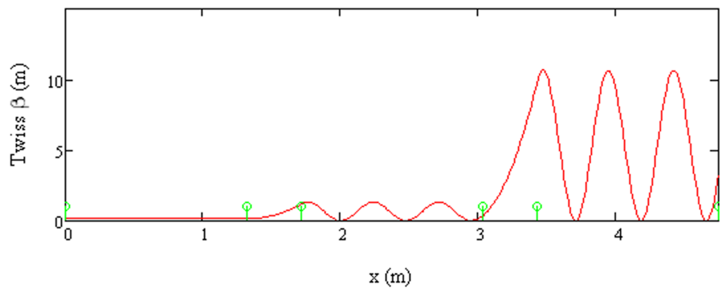

Figure 2: Behavior of the β_T function vs. z. The green signs indicate discontinuities between the undulators and the free space segments. The beam is matched to the constant envelope condition in the first undulator, then expands in the first drift and enters mismatched in the second undulator. The beam size then grows again at the second gap and is even more mismatched in the last undulator.

third undulator. Here, depending on the betatron oscillation phase, the beam expands again and enter the second undulator with an even stronger mismatch.

A strategy to avoid such a large mismatch at the third module, is to match to the invariant envelope solution Eq.1 at the entrance of the second module. This solution can be calculated by imposing that the Twiss parameters are the ones of Eq. 1 at the entrance of the second undulator. Indicating with G the operator propagating the beam from the optimized Twiss functions at the entrance of the undulator are therefore given by

$$\begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}_O = G^{-1} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}_M \tag{2}$$

In order to define the operator G, we introduce the transformation T that converts the 2×2 matrices propagating the orbit of a trajectory from $0 \rightarrow 1$

$$\begin{pmatrix} c & s \\ c' & s' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 = \begin{pmatrix} x \\ x' \end{pmatrix}_1 \tag{3}$$

into the 3×3 matrix that propagates the particle distribution Twiss parameters from $0 \rightarrow 1$=

$$T\left(\begin{pmatrix} c & s \\ c' & s' \end{pmatrix}\right) = \begin{pmatrix} c^2 & -2sc & s^2 \\ -cc' & sc' + s'c & -ss' \\ c'^2 & -2s'c' & s'^2 \end{pmatrix} \tag{4}$$

The operator G is therefore given by the operator T applied to the ray propagation matrix corresponding to the first undulator in the line and the drift separating the first from the second undulator, i.e.

$$G = T\left(\left(\begin{array}{cc} 1 & L_d \\ 0 & 1 \end{array}\right)\left(\begin{array}{cc} \cos(kL_u) & \frac{1}{k}\sin(kL_u) \\ -k\sin(kL_u) & \cos(kL_u) \end{array}\right)\right) \tag{5}$$

Where the parameter k is the focusing strength of the undulator,

$$k = \sqrt{h}\frac{\pi K}{\lambda_u \gamma} \tag{6}$$

L_u the undulator length and where h is the field rolling parameter, $h = 1$ for the Apple-X undulator in circular polarization. The Twiss parameters at the entrance of the line are therefore given by G^{-1} applied to the desired Twiss parameters at the entrance of the second undulator, i.e. those given by Eq. 1

$$T_0 = \left(\begin{array}{c} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{array}\right)_O = G^{-1}\left(\begin{array}{c} \frac{2\pi K}{\lambda_u \gamma} \\ 0 \\ \left(\frac{2\pi K}{\lambda_u \gamma}\right)^{-1} \end{array}\right) \tag{7}$$

The vector T_0 is given by

$$T_0 = \frac{1}{D}\left(\begin{array}{c} \frac{1}{k}\left[\cos(kL_u)^2 + N_p^2\right] \\ N_p N_m - \cos(kL_u)\sin(kL_u) \\ k\left[\sin(kL_u)^2 + N_m^2\right] \end{array}\right) \tag{8}$$

where

$$\begin{aligned} D &= \left[\cos(kL_u)^2 + \sin(kL_u)^2\right]^2 \\ N_p &= \sin(kL_u) + dk\cos(kL_u) \\ N_m &= \cos(kL_u) - dk\sin(kL_u) \end{aligned} \tag{9}$$

The solution corresponding to the input Twiss parameters T_0 is shown in Fig. 3. The resulting beam size is shown in Fig. 4. The matching condition to the invariant envelope in the undulator provides a beam size (RMS) of less than 400 μm, to be compared to 1 mm obtained when matching to the Twiss parameters Eq. 1.

The condition Eq.8 depends on the beam and undulator parameters. The beam energy and undulator strength have to be varied to reach the desired wavelength. In Fig. 5 we show the behavior of the β_T, α_T required at the undulator entrance to realize the condition Eq. 8.

The map gives an idea of the Twiss parameters range that should be ensured by the matching system, i.e. the quadrupoles that have to second the variation of parameters necessary to reach a specific wavelength. The calculation relies on a simplified model of the undulator that do not account for the undulator terminations and the magic fingers. For this reason we should keep safety margins

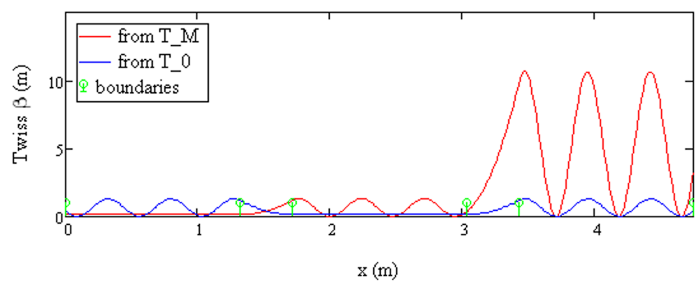

Figure 3: The blue line represents the of the β_T function vs. z starting from the optimized β_{TO} initial condition that realizes the invariant envelope in the second undulator instead that in the first (red line).

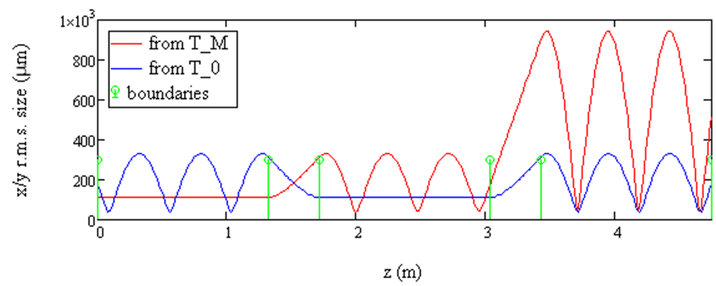

Figure 4: Beam sizes resulting from the propagation from the optimized β_{TO} initial condition (blue) or the β_{TM} initial condition (red).

of the limits shown in Fig. 5. The matching system should have the flexibility to span the range $0 \div 2$ for β_{TO} and $-4 \div 4$ for α_{TO}.

Besides the maps for the Twiss parameters, we are concerned about the transverse acceptance of the undulator system. The solution starting from the above conditions has a size plotted in Fig. 6.

In general, the wavelength tuning is obtained by varying the undulator strength at a fixed energy, and by selecting a few working points in energy. This ensures the possibility to quickly tune the wavelength without intervening on the accelerator or on the transfer line bringing the beam to the matching area. For SABINA we may consider four energy of operation that should ensure to cover the SABINA spectral range by tuning the undulator strength. If we consider the energies and Kparameter range given in Tab. 1.

Fig. 7 shows that these fixed energy points, combined to a variation of the undulator strength parameter ensure the possibility to tune the FEL wavelength in the range $10 - 100\ \mu$m required by SABINA. The four regions have a substantial overlap, but it must be noted that the FEL performances are expected

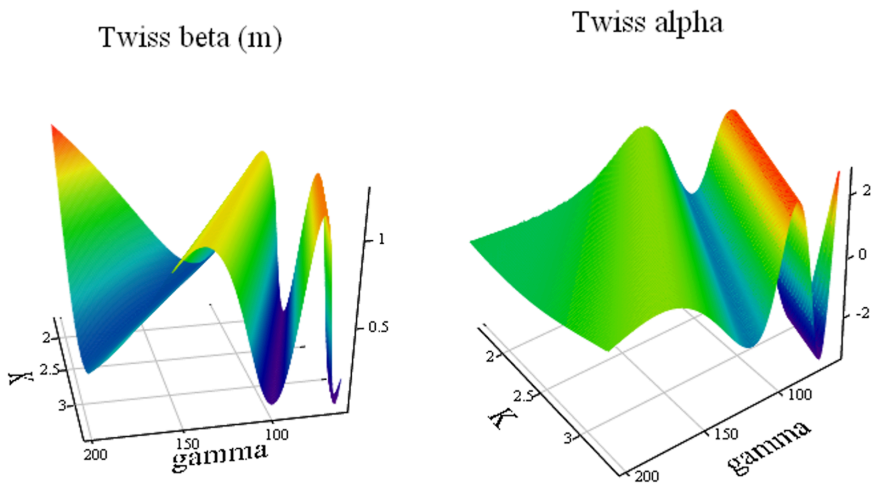

Figure 5: Parameter space of the Twiss functions required by the condition T_O as functions of the beam energy and undulator strength.

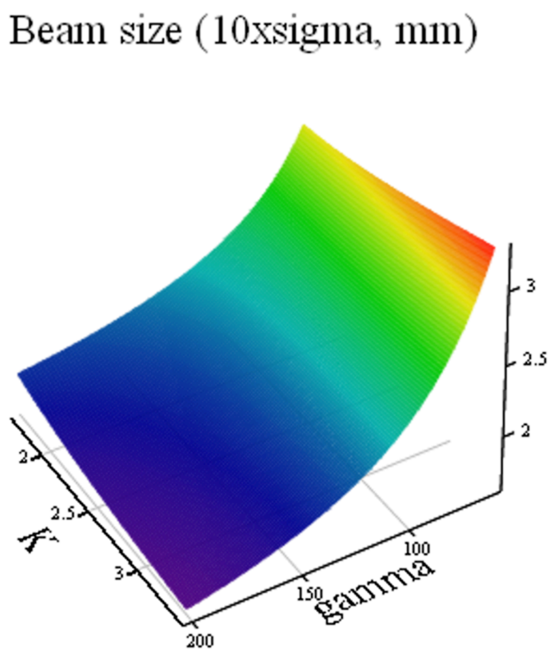

Figure 6: Beam size ($10 \times \sigma$) starting with the condition T_O as functions of the beam energy and undulator strength.

Table 1: List of beam energy working points and undulator strength K values covering the SABINA spectral range

Energy	K_{max}	K_{min}
30 MeV	3.4	1.7
45 MeV	3.4	1.7
65MeV	3.4	1.7
90 MeV	3.4	1.7

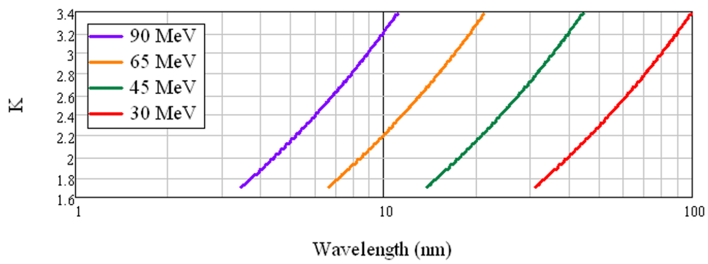

Figure 7: Wavelength range of SABINA covered by tuning the undulator parameter in a range $(1.7, 3.4)$ and by tuning the energy to the four working points indicated in the figure.

to be larger when K is close to the maximum value.

In Fig.8 we have repeated the calculations in Fig. 5 for the four specific energies listed in Tab. 1 preserving the color convention of Fig. 7.

The corresponding beam size, still calculated as 10× the RMS predicted size, is shown in Fig. 9. The size is less than two times the size of the vacuum chamber that has a diameter of 8 mm.

From the plot in Fig. 8 we observe that there are specific conditions that require a particularly small β_{TO} parameter. An analogous plot in logarithmic scale is shown in Fig. 10.

At low energy (30 MeV - red line) when K reaches the value $K = 3.2$ a β_{TO} of the order of $20\,\mu$m is required. Some mitigation of this issue can be achieved by reaching the same wavelength, but using a different beam energy. This certainly represents a complication from the tuning point of view, but could be a viable solution in case such low β_{TO} cannot be reached.

We finally analyze the tolerances on the input Twiss parameters. The matching quadrupoles have to be designed in order to ensure flexibility and to reach very low β_{TO} at the entrance of the undulator. A mismatch of the entrance parameters will reflect in a less than optimal transport, with a correlated growth of the beam size. In order to estimate the effect of a mismatch, we have calculated the relative beam size growth associated with detuned input Twiss parameters. The effect of the mismatch depends on the undulator strength and beam energy. We have therefore produced a plot similar to Fig. 9 where we calculate

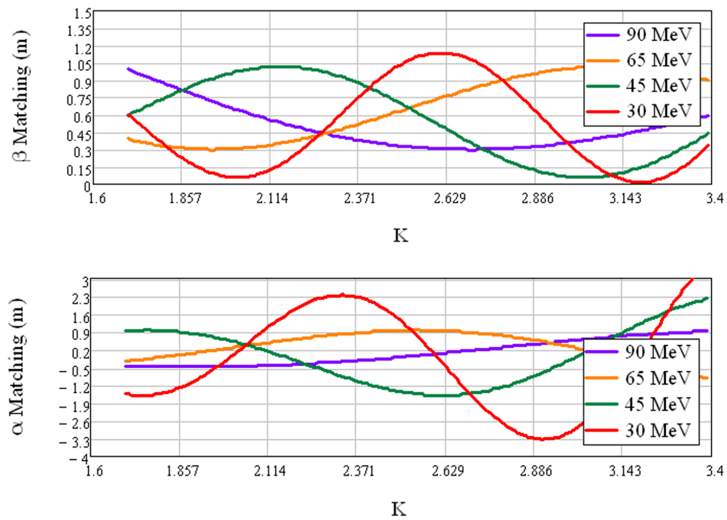

Figure 8: Twiss functions ranges of SABINA covered by tuning the undulator parameter in a range $(1.7, 3.4)$ and by tuning the energy to the four working points indicated in the figure.

the dependence on K and beam energy for the four working points of Tab. 1 induced by a relative variation of the input β_{TO}, $\delta\beta_{TO} = 20\%$ and a (absolute) variation $\delta\alpha_{TO} = 0.3$. The result of the analysis is shown in Fig. 12. We have considered two cases, for positive (top) and negative (bottom) variation of the Twiss functions. We report that the change of sign doesn't have a significative effect in the case of the α_{TO} (absolute) variation, while it is substantially worse for the negative $\delta\beta_{TO}$.

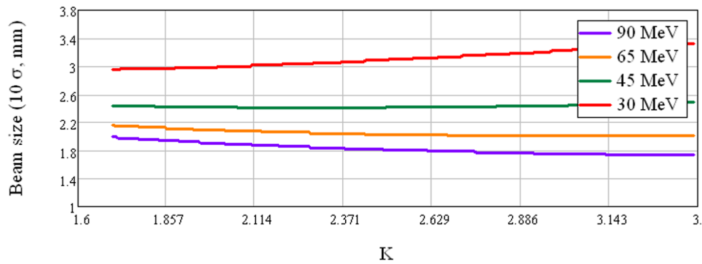

Figure 9: Beam size expected by tuning the undulator parameter in a range $(1.7, 3.4)$ and by tuning the energy to the four working points indicated in the figure when the beam is matched to T_O.

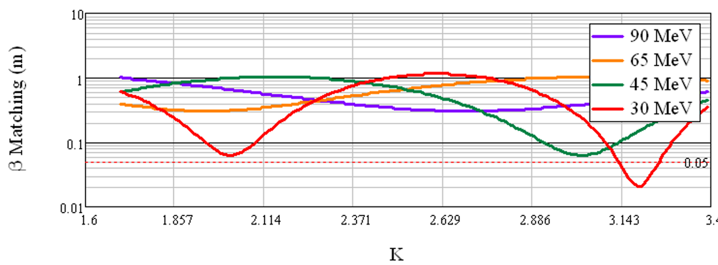

Figure 10: Same as in Fig. 8 plotted in *log* scale, in order to highlight the low β_T regions.

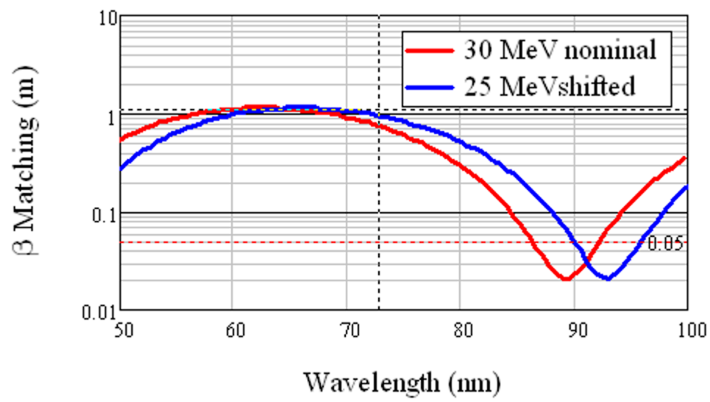

Figure 11: Effect of an energy detuning on the β_T function as a function of the wavelength.

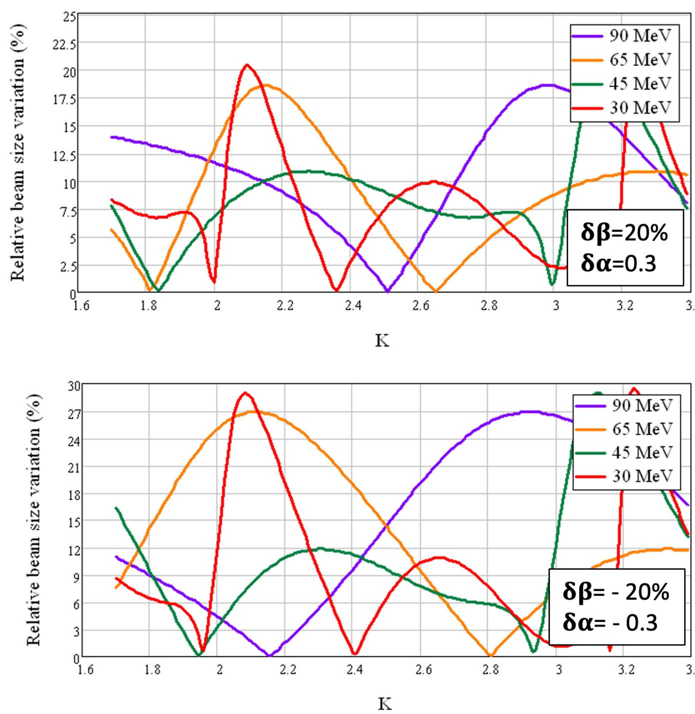

Figure 12: Relative increase of the beam size (%) for a variation of the input Twiss functions $\delta\beta_{TO} = 20\%$ and a (absolute) variation $\delta\alpha_{TO} = 0.3$.